Solutions to short-answer questions

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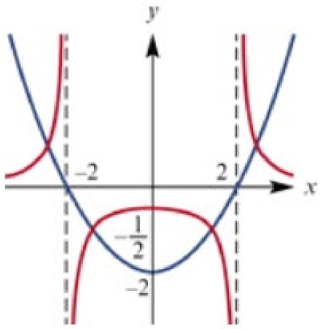

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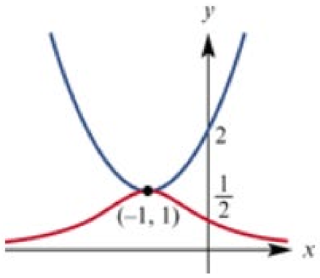

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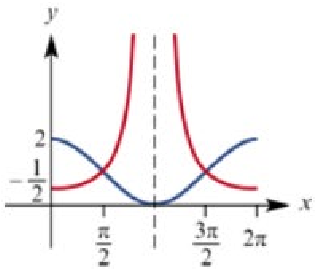

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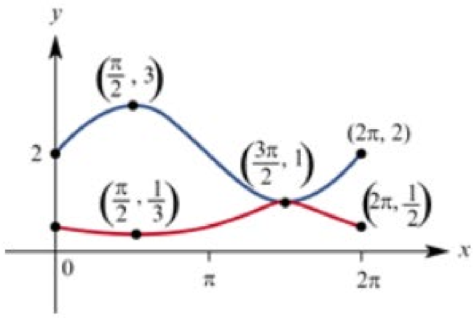
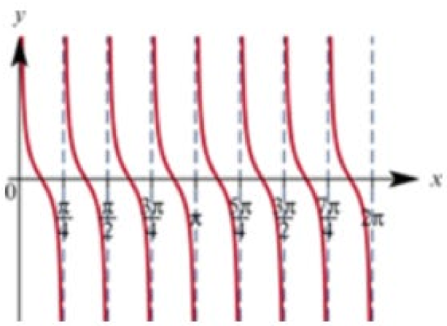


2 a



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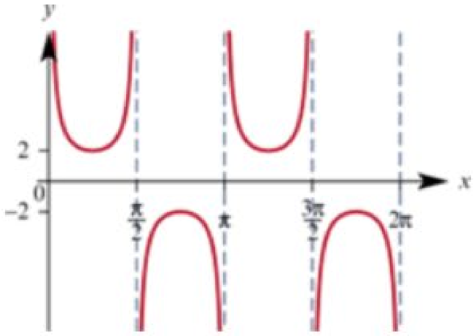

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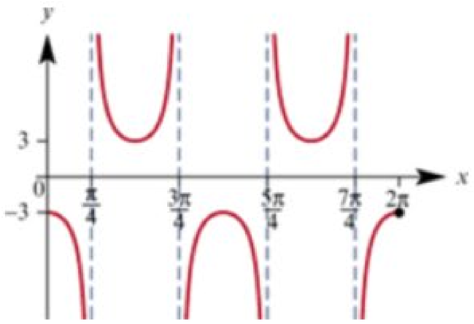

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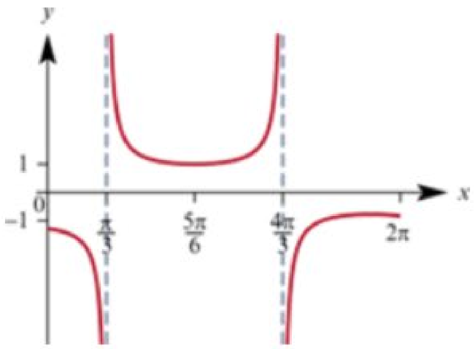

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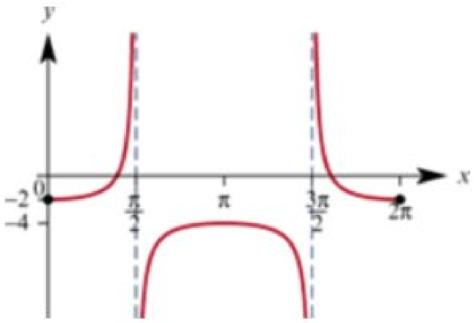

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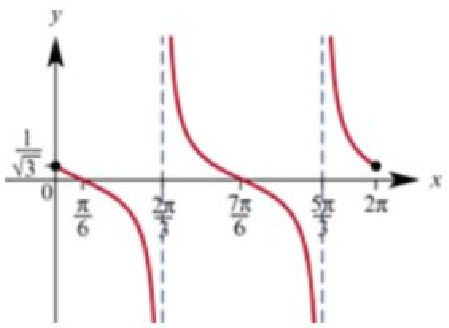


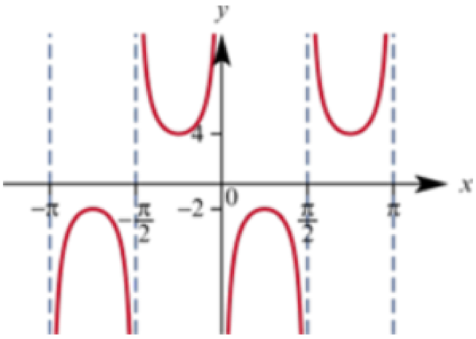
3

25A0 Reflection in the $x-$axis

25A0 Dilation of factor 3 from the $x-$axis

25A0 Dilation of factor $\frac{1}{2}$ from the $y-$axis

25A0 Translation 1 unit up



4 We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} QP &= RP \\ \sqrt{(x-2)^2 + (y-(-1))^2} &= \sqrt{(x-1)^2 + (y-2)^2} \\ (x-2)^2 + (y+1)^2 &= (x-1)^2 + (y-2)^2 \\ x^2 - 4x + 4 + y^2 + 2y + 1 &= x^2 - 2x + 1 + y^2 - 4y + 4 \\ 6y - 2x &= 0 \\ y &= \frac{x}{3}. \end{aligned}$$

Therefore, point P lies on the straight line with equation $y = \frac{x}{3}$.

5 We know that the point $P(x, y)$ satisfies,

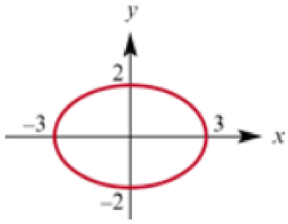
$$\begin{aligned} AP &= 5 \quad \text{This is a circle with centre } (3, 2) \text{ and radius 6 units.} \\ \sqrt{(x-3)^2 + (y-2)^2} &= 6 \\ (x-3)^2 + (y-2)^2 &= 6^2. \end{aligned}$$

6 We complete the square to find that

$$\begin{aligned} x^2 + 4x + y^2 - 8y &= 0 \\ [(x^2 + 4x + 4) - 4] + [(y^2 - 8y + 16) - 16] &= 0 \\ (x+2)^2 - 4 + (y-4)^2 - 16 &= 0 \\ (x+2)^2 + (y-4)^2 &= 20. \end{aligned}$$

This is the equation of a circle with centre $(-2, 4)$ and radius $\sqrt{20}$ units.

7 a



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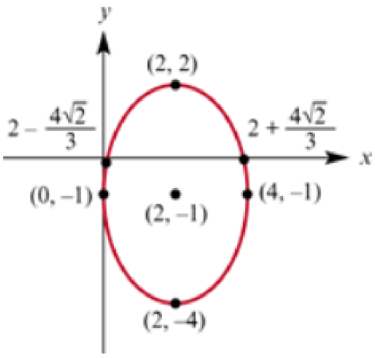


8 We complete the square to find that,

$$\begin{aligned} x^2+4x+2y^2&=0\\ (x^2+4x+4)-4+2y^2&=0\\ (x+2)^2+2y^2&=4\\ \frac{(x+2)^2}{4}+\frac{y^2}{2}&=1 \end{aligned}$$

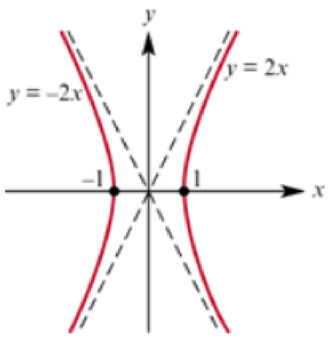
The centre is then $(-2,0)$. To find the x-intercepts we let $y=0$. Therefore,

$$\begin{aligned} \frac{(x+2)^2}{4}&=1\\ (x+2)^2&=4\\ x+2&=\pm 2\\ x&=-4,0 \end{aligned}$$

To find the y-intercepts we let $x=0$ (in the original equation). Therefore,

$$\begin{aligned} 2y^2&=0\\ y&=0. \end{aligned}$$

9 a



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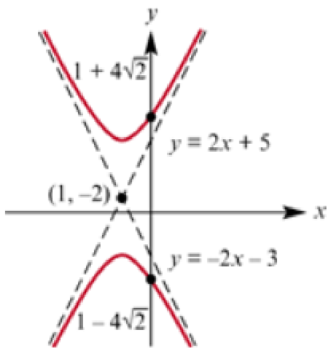


10 We know that the point $P(x,y)$ satisfies,

$$\begin{aligned} KP&=2MP\\ \sqrt{(x-(-2))^2+(y-5)^2}&=2\sqrt{(x-1)^2}\\ (x+2)^2+(y-5)^2&=4(x-1)^2\\ x^2+4x+4+y^2-10y+25&=4(x^2-2x+1)\\ x^2+4x+4+y^2-10y+25&=4x^2-8x+4\\ 3x^2-12x-y^2+10y-25&=0 \end{aligned}$$

Completing the square then gives,

$$\begin{aligned} 3(x^2-4x)-(y^2-10y)-25&=0\\ 3(x^2-4x+4-4)-(y^2-10y+25-25)-25&=0\\ 3((x-2)^2-4)-((y-5)^2+25)-25&=0\\ 3(x-2)^2-12-(y-5)^2-25-25&=0\\ 3(x-2)^2-(y-5)^2&=12\\ \frac{(x-2)^2}{4}-\frac{(y-5)^2}{12}&=1 \end{aligned}$$

Therefore, the set of points is a hyperbola with centre $(2, 5)$.

11a From the first equation we know that $t = \dfrac{x+1}{2}$. Substitute this into the second equation to get

$$\begin{aligned} y &= 6 - 4t \\ &= 6 - 4\frac{x+1}{2} \\ &= 6 - 2(x+1) \\ &= 6 - 2x - 2 \\ &= 4 - 2x \end{aligned}$$

We obtain straight line whose equation is $y = 4 - 2x$.

b We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x}{2} = \cos t \text{ and } \frac{y}{2} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

That is,

$$\begin{aligned} \frac{x^2}{2^2} + \frac{y^2}{2^2} &= 1 \\ x^2 + y^2 &= 2^2 \end{aligned}$$

which is a circle of radius 2 centred at the origin.

c We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x-1}{3} = \cos t \text{ and } \frac{y+1}{5} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+1}{5}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

That is,

$$\frac{(x-1)^2}{3^2} + \frac{(y+1)^2}{5^2} = 1,$$

which is an ellipse centred at the point $(1, -1)$.

d Since $x = \cos t$, we have,

$$\begin{aligned} y &= 3\sin^2 t - 2 \\ &= 3(1 - \cos^2 t) - 2 \\ &= 3 - 3\cos^2 t - 2 \\ &= 1 - 3\cos^2 t \\ &= 1 - 3x^2 \end{aligned}$$

Note that this does not give the entire parabola. Since $x = \cos t$, the domain will be $-1 \le x \le 1$. Therefore, the cartesian equation of the curve is

$$y = 1 - 3x^2, \text{ where } -1 \le x \le 1.$$

12a From the first equation we know that $t = x + 1$. Substitute this into the second equation to get

$$\begin{aligned} y &= 2t^2 - 1 \\ &= 2(x+1)^2 - 1. \end{aligned}$$

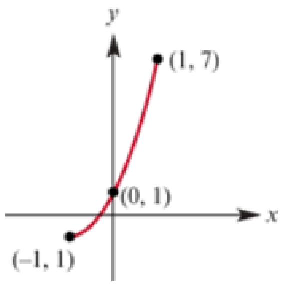
b Since $0 \le t \le 2$ and $x = t - 1$, we know that $-1 \le x \le 1$.

c The parabola has a minimum at $(-1, -1)$. It increases after this point. The maximum value of y is obtained when $x = 1$. Therefore,

$$y = 2(1+1)^2 - 1 = 7.$$

The range is the interval $-1 \le y \le 7$.

d We sketch the curve over the domain $-1 \le x \le 1$.



13 We have

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$
$$= 2\cos 3\pi/4 \qquad\qquad = 2\sin 3\pi/4$$
$$= -\sqrt{2} \qquad\qquad = \sqrt{2}$$

so that the cartesian coordinates are $(-\sqrt{2}, \sqrt{2})$.

14 $r = \sqrt{2^2 + (-2\sqrt{3})^2} = \sqrt{16} = 4$

$\theta = \tan^{1} \dfrac{-2\sqrt{3}}{2} = \tan^{-1} -\sqrt{3} = -\dfrac{\pi}{3}$

The point has polar coordinates $(4, -\frac{\pi}{3})$. We could also let $r = -4$ and add π to the found angle, giving coordinate $(-4, \frac{2\pi}{3})$.

15 Since $x = r\cos\theta$ and $y = r\sin\theta$ the equation becomes,

$$2x + 3y = 5$$
$$2r\cos\theta + 3r\sin\theta = 5$$
$$r(2\cos\theta + 3\sin\theta) = 5$$

Therefore the polar equation is,

$$r = \frac{5}{2\cos\theta + 3\sin\theta}.$$

16 The trick here is to first multiply both sides of the expression through by r to get

$$r^2 = 6r\sin\theta \qquad (1)$$

Since $r^2 = x^2 + y^2$ and $r\sin\theta = y$, equation (1) becomes,

$$x^2 + y^2 = 6y$$
$$x^2 + y^2 - 6y = 0$$
$$x^2 + (y^2 - 6y + 9) - 9 = 0 \qquad \text{(completing the square)}$$
$$x^2 + (y-3)^2 = 9.$$

This is a circle whose centre is $(0, 3)$ and whose radius is 3, as required.

Solutions to multiple-choice questions

1 **B** The graph will have two vertical asymptotes provided that the denominator has two x-intercepts. Therefore the discriminant of the quadratic must satisfy,

$$\Delta > 0$$
$$b^2 - 4ac > 0$$
$$64 - 4(1)k > 0$$
$$64 - 4k > 0$$
$$k < 16.$$

2 **B** It is a graph of $y = \sec x$ transformed. It is reflected in the $x-$ axis, dilated by factor 2 from the $x-$ axis and translated 1 unit in the positive direction of the $y-$ axis.

3 **A** We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} AP &= BP \\ \sqrt{(x-2)^2+(y+5)^2} &= \sqrt{(x+4)^2+(y-1)^2} \\ (x-2)^2+(y+5)^2 &= (x+4)^2+(y-1)^2 \\ x^2-4x+4+y^2+10y+25 & \\ &= x^2+8x+16+y^2-2y+1 \\ y &= x-1 \end{aligned}$$

Therefore, the set of points is a straight line with equation $y = x - 1$. Alternatively, one could also just find the perpendicular bisector of line AB. This will give the same equation for about the same effort.

4 **D** One can answer this question either by reasoning geometrically, or by finding the equation of the parabola. Suppose MP is the perpendicular distance from the line $y = -2$ to the point P. We know that the point $P(x, y)$ satisfies,

$$\begin{aligned} FP &= MP \\ \sqrt{x^2+(y-2)^2} &= \sqrt{(y-(-2))^2} \\ x^2+(y-2)^2 &= (y+2)^2 \\ x^2+y^2-4y+4 &= y^2+4y+4 \\ x^2 &= 8y \\ y &= \frac{x^2}{8}. \end{aligned}$$

Clearly **A**,**B** and **C** are true. The point $(2, 1)$ does not lie on the parabola since when $x = 2$,

$$y = \frac{x^2}{8} = \frac{2^2}{8} \neq 1.$$

The point $(4, 2)$ does lie on the parabola since when $x = 4$,

$$y = \frac{x^2}{8} = \frac{4^2}{8} = 2.$$

5 **C** Since the x-intercepts are $x = \pm 3$ and the y-intercepts are $y = \pm 2$ the equation must be

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

which clearly corresponds to item C.

6 **D** The hyperbola is centred at the point $(2, 0)$. This means that we can exclude options **A**,**C** and **E**, each of which are centred at the point $(-2, 0)$. The x-intercepts of the hyperbola occur when $x = -7$ and $x = 11$. We let $y = 0$ in option **B** and **D**, and see that only option **D** has the correct intercepts.

7 **C** The graph of

$$\frac{y^2}{9} - \frac{x^2}{4} = 1$$

is centred at the point $(0, 0)$. If we translate this by 3 units to the left and 2 units up we obtain the given equation. It will now be centred at the point $(-3, 2)$.

8 **C** We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This means that

$$\frac{x-1}{4} = \cos t \text{ and } \frac{y+1}{2} = \sin t.$$

We then use the Pythagorean identity to show that

$$\left(\frac{x-1}{4}\right)^2 + \left(\frac{y+1}{2}\right)^2 = \cos^2 t + \sin^2 t = 1.$$

That is,

$$\frac{(x-1)^2}{4^2}+\frac{(y+1)^2}{2^2}=1.$$

To find the x-intercepts, we let $y=0$. Solving for x gives,

$$\begin{aligned}\frac{(x-1)^2}{4^2}+\frac{(0+1)^2}{2^2}&=1\\ \frac{(x-1)^2}{4^2}+\frac{1}{4}&=1\\ \frac{(x-1)^2}{4^2}&=\frac{3}{4}\\ (x-1)^2&=12\\ x-1&=\pm\sqrt{12}\\ x&=1\pm 2\sqrt{3}\end{aligned}$$

9 **E** **Option A:** These points are in quadrants 1 and 2 respectively and so cannot represent the same point.

Option B: These are located on the y-axis, but on opposite sides.

Option C: These points are in quadrants 1 and 4 respectively so cannot represent the same point.

Option D: These points are in quadrants 1 and 3 respectively so cannot represent the same point.

Option E: These coordinates do represent the same point. Recall that the coordinate $(-1, 7\pi/6)$ means that we locate direction $7\pi/6$, then move 1 unit in the opposite direction. This is the same as moving 1 unit in the direction $\pi/6$.

10 **B** The trick is to multiply both sides of the equation through by r. This gives,

$$\begin{aligned}r^2&=r+r\cos\theta\\ x^2+y^2&=r+x\\ x^2+y^2-x&=r\\ x^2+y^2-x&=\sqrt{x^2+y^2}\\ (x^2+y^2-x)^2&=x^2+y^2,\end{aligned}$$

as required.

Solutions to extended-response questions

1 **a** We know that the point $P(x, y)$ satisfies,

$$\begin{aligned}AP&=BP\\ \sqrt{x^2+(y-3)^2}&=\sqrt{(x-6)^2+y^2}\\ x^2+(y-3)^2&=(x-6)^2+y^2\\ x^2+y^2-6y+9&=x^2-12x+36+y^2\\ -6y+9&=-12x+36\\ y&=2x-\frac{9}{2}\end{aligned}$$

Therefore, the set of points is a straight line with equation $y=2x-\frac{9}{2}$.

b We know that the point $P(x, y)$ satisfies,

$$\begin{aligned}AP&=2BP\\ \sqrt{x^2+(y-3)^2}&=2\sqrt{(x-6)^2+y^2}\\ x^2+(y-3)^2&=4[(x-6)^2+y^2]\\ x^2+y^2-6y+9&=4[x^2-12x+36+y^2]\\ 3x^2-48x+3y^2+6y+135&=0\end{aligned}$$

Completing the square then gives,

$$\begin{aligned}3x^2 - 48x + 3y^2 + 6y + 135 &= 0\\ 3(x^2 - 16x) + 3(y^2 + 2y) + 135 &= 0\\ 3[(x^2 - 16x + 64) - 64] + 3[(y^2 + 2y + 1) - 1] + 135 &= 0\\ 3[(x-8)^2 - 64] + 3[(y+1)^2 - 1] + 135 &= 0\\ 3(x-8)^2 + 3(y+1)^2 &= 60\\ (x-8)^2 + (y+1)^2 &= 20\end{aligned}$$

This defines a circle with centre $(8, -1)$ and radius $\sqrt{20}$.

2 a Suppose MP is the perpendicular distance from the line $y = -2$ to the point P. We know that the point $P(x, y)$ satisfies,

$$\begin{aligned}FP &= MP\\ \sqrt{x^2 + (y-4)^2} &= \sqrt{(y - (-2))^2}\\ x^2 + (y-4)^2 &= (y+2)^2\\ x^2 + y^2 - 8y + 16 &= y^2 + 4y + 4\\ 12y &= x^2 + 12\\ y &= \frac{x^2}{12} + 1.\end{aligned}$$

Therefore, the set of points is a parabola.

b Suppose MP is the perpendicular distance from the line $y = -2$ to the point P. We know that the point $P(x, y)$ satisfies,

$$\begin{aligned}FP &= \frac{1}{2}MP\\ \sqrt{x^2 + (y-4)^2} &= \frac{1}{2}\sqrt{(y - (-2))^2}\\ x^2 + (y-4)^2 &= \frac{1}{4}(y+2)^2\\ 4[x^2 + (y-4)^2] &= (y+2)^2\\ 4(x^2 + y^2 - 8y + 16) &= y^2 + 4y + 4\\ 4x^2 + 4y^2 - 32y + 64 &= y^2 + 4y + 4\\ 4x^2 + 3y^2 - 36y + 60 &= 0\end{aligned}$$

Completing the square then gives,

$$\begin{aligned}4x^2 + 3y^2 - 36y + 60 &= 0\\ 4x^2 + 3[y^2 - 12y + 20] &= 0\\ 4x^2 + 3[(y^2 - 12y + 36) - 36 + 20] &= 0\\ 4x^2 + 3[(y-6)^2 - 16] &= 0\\ 4x^2 + 3(y-6)^2 &= 48\\ \frac{x^2}{12} + \frac{(y-6)^2}{16} &= 1\end{aligned}$$

Therefore, the set of points is an ellipse.

c Suppose MP is the perpendicular distance from the line $y = -2$ to the point P. We know that the point $P(x, y)$ satisfies,

$$\begin{aligned}FP &= 2MP\\ \sqrt{x^2 + (y-4)^2} &= 2\sqrt{(y - (-2))^2}\\ x^2 + (y-4)^2 &= 4(y+2)^2\\ x^2 + (y-4)^2 &= 4(y+2)^2\\ x^2 + y^2 - 8y + 16 &= 4(y^2 + 4y + 4)\\ x^2 + y^2 - 8y + 16 &= 4y^2 + 16y + 16\\ x^2 - 3y^2 - 24y &= 0\end{aligned}$$

Completing the square then gives,

$$\begin{aligned} x^2 - 3y^2 - 24y &= 0 \\ x^2 - 3[y^2 + 8y] &= 0 \\ x^2 - 3[y^2 + 8y + 16 - 16] &= 0 \\ x^2 - 3[(y+4)^2 - 16] &= 0 \\ 3(y+4)^2 - x^2 &= 48 \\ \frac{(y+4)^2}{16} - \frac{x^2}{48} &= 1 \end{aligned}$$

Therefore, the set of points is a hyperbola.

3 **a** Since $x = 10t$, we know that $t = \frac{x}{10}$. We substitute this into the second equation to give

$$\begin{aligned} y &= 20t - 5t^2 \\ &= 20\left(\frac{x}{10}\right) - 5\left(\frac{x}{10}\right)^2 \\ &= 2x - 5\frac{x^2}{100} \\ &= 2x - \frac{x^2}{20} \end{aligned}$$

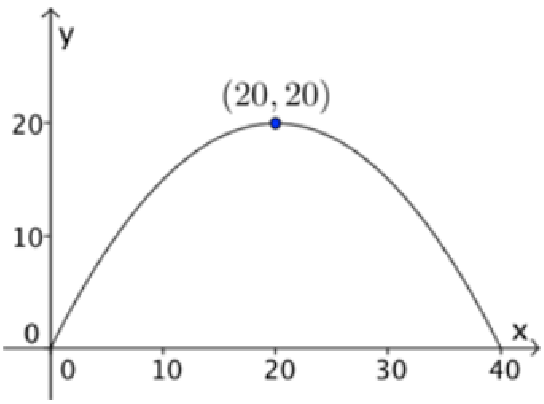
It will also help later if we consider the factorised expression. That is,

$$y = \frac{1}{20}x(40 - x).$$

b The equation of the ball's path is

$$y = \frac{1}{20}x(40 - x).$$

We note that the x-intercepts are $x = 0, 40$. The turning point will be located half-way between at $x = 20$. When $x = 20$, we find that $y = 20$. The graph of the ball's path is shown below.

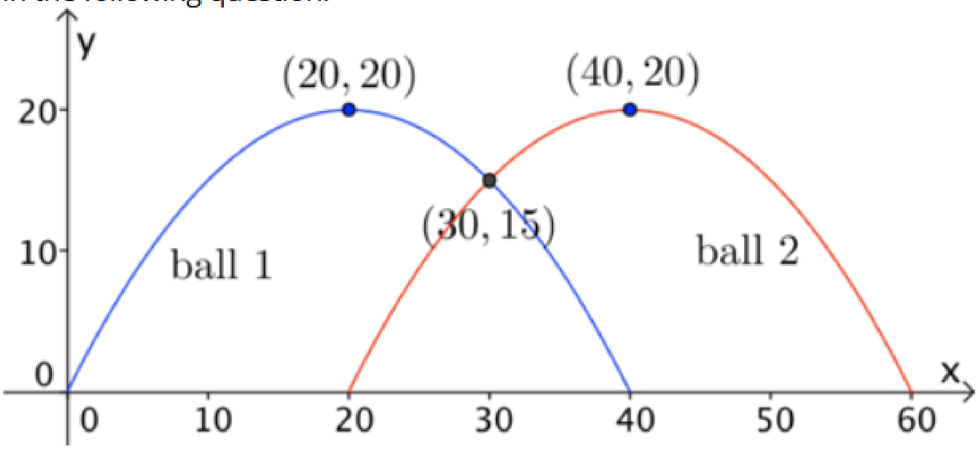


c The maximum height reached by the balls is 20 metres, and occurs when $x = 20$.

d Since $x = 10t$, we know that $t = \dfrac{60 - x}{10}$. We substitute this into the second equation to give

$$\begin{aligned} y &= 20t - 5t^2 \\ &= 20\left(\frac{60-x}{10}\right) - 5\left(\frac{60-x}{10}\right)^2 \\ &= 120 - 2x - 5\frac{(60-x)^2}{100} \\ &= 120 - 2x - \frac{(60-x)^2}{20} \\ &= 120 - 2x - \frac{(3600 - 120x + x^2)^2}{20} \\ &= 120 - 2x - 180 + 6x - \frac{x^2}{20} \\ &= 4x - 60 - \frac{x^2}{20} \\ &= -\frac{1}{20}(x^2 - 80x + 1200) \\ &= -\frac{1}{20}(x-20)(x-60) \end{aligned}$$

e The second ball's path has been included on the diagram below. The point of intersection has been identified in the following question.



f To find where the paths meet, we solve the following pair of equations simultaneously (or using your calculator),

$$y = -\frac{1}{20}(x-20)(x-60) \quad (1)$$

$$y = \frac{1}{20}x(40-x) \quad (2)$$

This gives a solution of $x = 30$ and $y = 15$.

g Note: just because the paths cross does *not* automatically mean that the balls collide. For this to happen, they must be at the same point at the same *time*. For the first ball, when $x = 30$, we find that

$$t = \frac{x}{10} = \frac{30}{10} = 3.$$

For the second ball, when $x = 30$, we find that

$$t = \frac{60-x}{10} = \frac{60-30}{10} = 3.$$

So the balls are at the same position at the same time. Therefore, they collide.

4 The ladder is initially vertical with its midpoint located 3 metres up the wall at coordinate $(0, 3)$. The ladder comes to a rest lying horizontally. Its midpoint is located 3 metres to the right of the wall at coordinate $(3, 0)$. So if the midpoint is to move along a circular path then it must be along the circle

$$x^2 + y^2 = 3^2 \quad (1)$$

To check that this is indeed true, we suppose that the ladder is t units from the base of the wall. Then by

Pythagoras' theorem, the ladder reaches

$$s = \sqrt{6^2 - t^2} = \sqrt{36 - t^2}$$

units up the wall. The midpoint of the ladder will then be

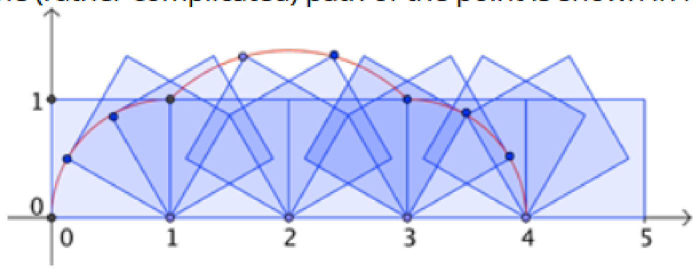
$$P\left(\frac{t}{2}, \frac{\sqrt{36 - t^2}}{2}\right).$$

We just need to check that this point lies on the circle whose equation is (1). Indeed,

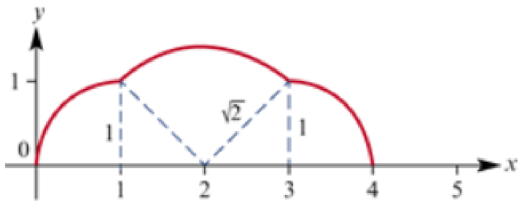
$$\begin{aligned} x^2 + y^2 &= \left(\frac{t}{2}\right)^2 + \left(\frac{\sqrt{36 - t^2}}{2}\right)^2 \\ &= \frac{t^2}{4} + \frac{36 - t^2}{4} \\ &= \frac{36}{4} \\ &= 9 \\ &= 3^2 \end{aligned}$$

Therefore point P lies on the circle whose equation is $x^2 + y^2 = 3^2$.

5 a The (rather complicated) path of the point is shown in red below.



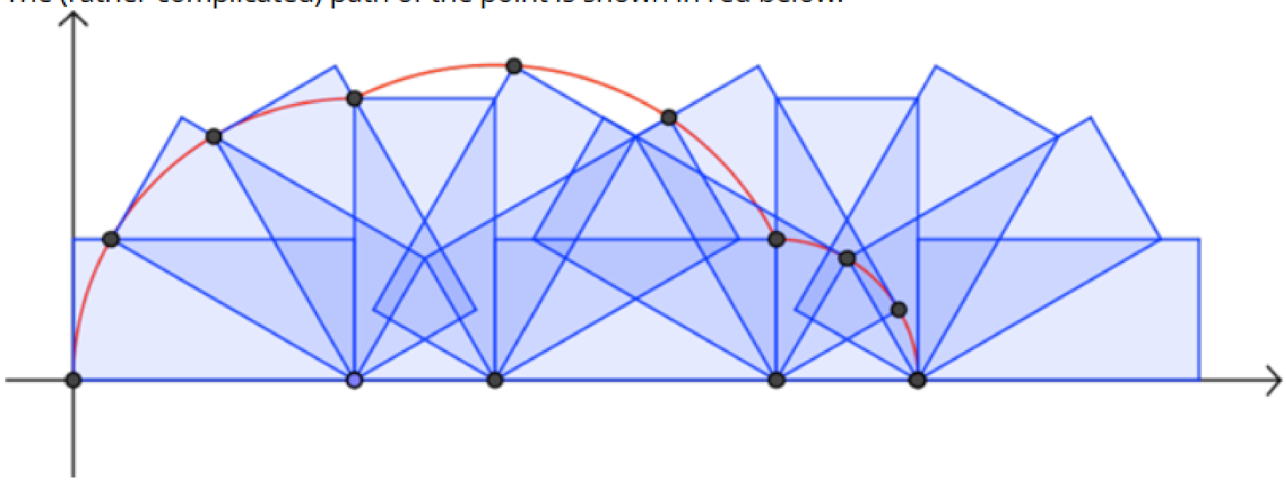
The picture is a little more clear when the box is hidden. The path consists of a quarter circle of of radius 1, another quarter circle of radius $\sqrt{2}$ (the diagonal length) and then another quarter circle of length 1.



b The total distance covered by point P will be

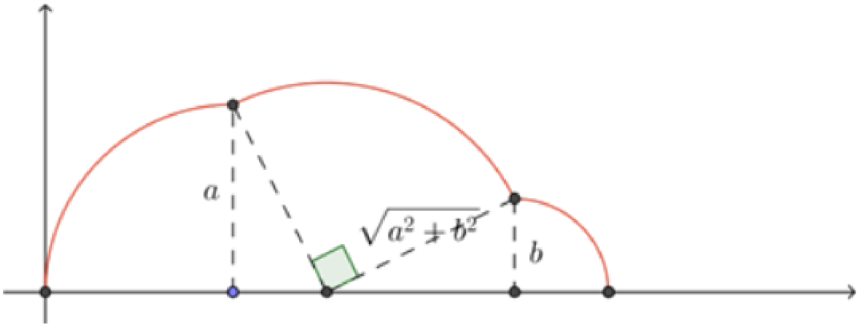
$$D = \frac{1}{4}(2\pi \cdot 1 + 2\pi \cdot \sqrt{2} + 2\pi \cdot 1) = \frac{1}{2}(2\pi + \pi\sqrt{2}).$$

c The (rather complicated) path of the point is shown in red below.



The picture is a little more clear when the box is hidden. The path consists of a quarter circle of of radius a, another quarter circle of radius $\sqrt{a^2 + b^2}$ (the diagonal length) and then another quarter circle of length b.

d



The total distance covered by point P will then be

$$D = \frac{1}{4}(2\pi a + 2\pi \cdot \sqrt{a^2+b^2} + 2\pi b) = \frac{\pi}{2}(a + \sqrt{a^2+b^2} + b).$$

The area consists of three quarter circles and two triangles. The total area will then be.

$$\begin{aligned} A &= \frac{1}{4}(\pi a^2 + \pi(\sqrt{a^2+b^2})^2 + \pi b^2) + 2 \times \frac{1}{2}ab \\ &= \frac{1}{4}(2\pi a^2 + 2\pi b^2) + ab \\ &= \frac{\pi}{2}(a^2+b^2) + ab. \end{aligned}$$